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LETTER TO THE EDITOR

A Fokker–Planck equation for the fluctuations of the heat flux

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Abstract. The problem of heat fluctuations around the steady state is considered in the framework of extended irreversible thermodynamics. A new version using a Fokker–Planck description is carried out. Averages for the fluctuations around the steady state and correlation functions for these fluctuations are calculated.

Heat fluctuations around the steady state have been treated by Landau and Lifshitz (1971) and Fox and Uhlenbeck (1970), making use of the Onsager and Machlup (1953) ideas. In a previous paper Jou and Rubí (1979) have carried out a study of this problem when effects due to the introduction of a relaxation time for the propagation of thermal signals are considered. Extended irreversible thermodynamics (Müller 1967, Lambermont and Lebon 1973, Lebon *et al* 1979) was used in order to introduce such effects in a thermodynamical scheme. Like some other authors (Keizer 1976), we employed the Einstein formula for the probability of a fluctuation, and in this way it was possible to calculate the correlation function for the fluctuations of the heat flux. A generalisation to thermoviscous fluids has been carried out by Jou *et al* (1980) and another to electrical conductors by Jou and Llebot (1980). Also, the previous results have been corroborated in the framework of the Onsager–Machlup function (Jou and Casas-Vázquez 1980).

The purpose of this Letter is to describe the fluctuations around the steady state by means of the Fokker–Planck formalism. Such a formalism has recently been utilised by Enz (1978) in order to describe Navier–Stokes fluids.

To study isotropic rigid heat conductors without the local equilibrium hypothesis, we can use a generalised Gibbs equation of the form

$$\mathrm{d}s = T^{-1}\,\mathrm{d}u + T^{-1}v\alpha \boldsymbol{q}\cdot\mathrm{d}\boldsymbol{q} \tag{1}$$

which introduces the heat flux $\boldsymbol{q}$ as a new independent variable. In (1) s, u and v are respectively the specific entropy, the specific internal energy and the specific volume, T is the absolute temperature and α a parameter given through the expression

$$(\partial s/\partial \boldsymbol{q})_u = T^{-1}v\alpha\boldsymbol{q}. \tag{2}$$

From (1) it is possible to obtain a generalised constitutive equation for the heat flux in the form (Lambermont and Lebon 1973)

$$\dot{\boldsymbol{q}} = -\tau^{-1}(\boldsymbol{q} + \lambda\nabla T) \tag{3}$$

where λ is the thermal conductivity, τ a relaxation time equal to $-\alpha\lambda T$ and an upper dot stands for substantial derivation. In the limit when $\tau \to 0$, one recovers the classical expression $\boldsymbol{q} = -\lambda \nabla T$.

In order to take into account the effects of the remaining faster variables on the evolution of the heat flux, we have generalised (3) (Jou and Rubí 1979) by including a Gaussian stochastic noise f. In this way the fluctuations around an equilibrium state can be described by means of the classical Langevin equation

$$\dot{\boldsymbol{p}} = -\tau^{-1}\boldsymbol{p} + \boldsymbol{f} \tag{4}$$

where $\boldsymbol{p} = \boldsymbol{q} - \boldsymbol{q}^{\mathrm{s}}$, $\boldsymbol{q}^{\mathrm{s}}$ being the heat flux at the steady state. Moreover, the stochastic term f satisfies a fluctuation–dissipation theorem

$$\langle f_i(t) f_j(t+t')\rangle = 2k\lambda T^2 \tau^{-2} \delta(t')\delta_{ij} \tag{5}$$

where k is the Boltzmann constant.

Sometimes, we can solve Langevin equations by converting them into Fokker–Planck equations and in this way calculate the averages by means of a probability distribution. Equation (4) is equivalent to the Fokker–Planck equation (Haken 1978)

$$\dot{f} = \mathrm{d}(\tau^{-1}pf)/\mathrm{d}p + (Q/2)\,\mathrm{d}^2 f/\mathrm{d}p^2 \tag{6}$$

where for simplicity we have taken into account unidimensional heat propagation and Q is the diffusion coefficient, which in our case is

$$Q = 2k\lambda T^2 \tau^{-2}. \tag{7}$$

Time-dependent solutions of the Fokker–Planck equation can be obtained by means of different techniques, for instance by path integrals (Haken 1976). In our case (6) admits a trivial solution

$$f(p, t) = N(t) \exp(-p^2/a + 2bp/a) \tag{8}$$

where $N(t)$ is a time-dependent normalisation factor and a and b time-dependent functions which are given by

$$\begin{aligned} &a(t) = 2k\lambda T^2 \tau^{-1}(1 - \exp(-2t/\tau) + a_0 \exp(-2t/\tau)), \\ &b(t) = b_0 \exp(-t/\tau), \qquad N(t) = (\pi a)^{-1/2} \exp(-b^2/a), \end{aligned} \tag{9}$$

where $a_0 = a(0)$ and $b_0 = b(0)$. From (8) and (9), the average for the fluctuations around the steady state exhibits a relaxative behaviour and reads

$$\langle p(t)\rangle = b_0 \exp(-t/\tau), \tag{10}$$

which in the limit $\tau \to 0$ (classical case) vanishes. In the same way, one can calculate the two-time correlation function

$$\langle p(0)p(t)\rangle = k\lambda T^2 \tau^{-1} \exp(-t/\tau), \tag{11}$$

which coincides with our previous result (Jou and Rubí 1979).

Since the deterministic term in the Langevin equation (4) is linear, the Fokker–Planck equation admits a trivial solution and in this way averages of heat fluctuation can be calculated. Such an equation is of great value in the computation of time correlation functions without using a complete molecular description and, therefore, in the treatment of heat conducting processes, we hope that it may be able to be used to apply the linear response theories of Kubo and Mori (Zwanzig 1973) to such systems.

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